AMS Foundation Exam - Part A, May 2017

Name: ______________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Let n be an odd, positive integer and let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Show that $\mathbf{A}$ has a real eigenvalue.

2. Let S and T be bases for a two-dimensional vector space V and let A and B be mappings from $V \to V$. Suppose that

$$[A]_S = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } [B]_T = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Is $A = B$? Justify your answer.

3. Find bases for, and the dimensions of, the four fundamental subspaces of

$$\mathbf{M} = \begin{bmatrix} 2 & 6 & 10 & 0 & 14 \\ 1 & 3 & 5 & 1 & 9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

4. Let $P_2(t)$ be the vector space of polynomials with degree ≤ 2 with an inner product given by $\langle f, g\rangle = \int_0^1 \mathrm{d}t f(t)g(t)$. Determine a basis for the subspace of $P_2(t)$ orthogonal to $h(t) = 3t+1$.

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. (a) Evaluate $\lim_{x\to 0} (\cos(x))^{1/x^2}$.

   (b) Let $f(x)$ be singular at $x = 0$, but otherwise differentiable on $\mathbb{R}$. If

$$\lim_{\varepsilon\to 0^+} \left[\int_{-2}^{-\varepsilon} f(x)\mathrm{d}x + \int_{\varepsilon}^{2} f(x)\mathrm{d}x\right] = 5$$

   but $\lim_{\varepsilon\to 0^+} \left[\int_{-2}^{-\varepsilon} f(x)\mathrm{d}x + \int_{2\varepsilon}^{2} f(x)\mathrm{d}x\right]$ does not exist, what can you determine about $\int_{-2}^{2} f(x)\mathrm{d}x$? Explain.

2. For $n \in \{1, 2, \ldots\}$, prove that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2 \sin(x)\right) = \left(x^2 - n(n-1)\right) \sin\left(x + \frac{n\pi}{2}\right) - 2nx \cos\left(x + \frac{n\pi}{2}\right).$$

3. Let $R \subset \mathbb{R}^2$ be the region defined by $\dfrac{-2}{x} < y < \dfrac{1}{x}$, $1 < x^2 - y^2 < 3$, and $x > 0$. Evaluate

$$\iint_R \mathrm{d}A(x,y) \frac{2(x^2+y^2)\sin(3-xy)}{x^2-y^2+2}.$$

Hint: Choose a coordinate system that makes R simple.

4. Consider $P = \int_C \vec{F} \cdot \mathrm{d}\vec{r}$ where

$$\vec{F}(x, y) = (x + 2y)\mathbf{i} + (x^2 + 1)\mathbf{j}$$

and C is the path from $(0, 0)$ to $(0, 1)$ following $x = a(y - y^2)$ for some $a > 0$. Determine the value of a that minimizes P.